Paratopological groups. Basic facts

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Chapter 1

Paratopological groups. Basic facts

1.1 Basic definitions

In classic abstract algebra we can distinguish many different objects (semigroups, monoids, groups). Here we will equip them with topologies. In this section we will introduce few basic definitions about groups. First we will present definitions of left and right topological semigroups.

Definition 1. *A right topological semigroup consists of a semigroup S and a topology τ on S such that for all $a \in S$, the right action ϱ_a of a on S is a continuous mapping of the space S to itself.*

Definition 2. *A left topological semigroup consists of a semigroup S and a topology τ on the set S such that for all $a \in S$, the left action λ_a of a on S is a continuous mapping of the space S to itself.*

Definition 3. *A semitopological semigroup is a right topological semigroup which is also a left topological semigroup.*

Now we define some kinds of topological groups.

Definition 4. *A left (right) topological group is a left (right) topological semigroup whose underlying semigroup is a group, and a semitopological group is a left topological group which is also a right topological group.*

Definition 5. *A paratopological group G is a group G with a topology on the set G that makes the multiplication mapping $G \times G \to G$ continuous, when $G \times G$ is given the product topology.*

Definition 6. *For a group G, the inverse mapping $In\colon G \to G$ is defined by the rule $In(x) = x^{-1}$, for each $x \in G$. A semitopological group with continuous inverse is called a quasitopological group.*

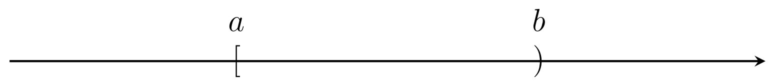
Definition 7. *A topological group G is a paratopological group G such that the inverse mapping $In\colon G \to G$ is continuous.*

Fact 1. *An easy verification shows that G is a topological group if and only if the mapping $(x, y) \mapsto xy^{-1}$ of $G \times G$ to G is continuous.*

In our next considerations we will focus on **paratopological groups**. In next section we will present some important fact about this structure.

1.2 Examples

1. Consider the topology on $\mathbb{R}$ with the base $\mathcal{B}$ consisting of the sets $[a, b) = \{x \in R\colon a \leqslant x < b\}$, where $a, b \in \mathbb{R}$ and $a < b$. For any a and b, the interval $[a, b)$ is clopen in $\mathbb{R}$



With this topology, and the natural addition in the role of multiplication, $\mathbb{R}$ is a paratopological group and, therefore,a topological semigroup. However, $(\mathbb{R}, \tau)$ is not a topological group since the inverse operation $x \mapsto -x$ is discontinuous. This paratopological group is called the **Sorgenfrey line**.
Sorgenfrey line has many interesting properties

- Sorgenfrey line is finer (has more open sets) than the standard topology on the real numbers (which is generated by the open intervals). The reason is that every open interval can be written as a countably infinite union of half-open intervals.
- Any compact subset of $(\mathbb{R}, \tau)$ must be an at most countable set.
- It is a perfectly normal Hausdorff space.
- It is first-countable and separable, but not second-countable.
- is a Baire space.

2. Another example is also connected with the Sorgenfrey line.
Let $G = S^2$ where S is the Sorgenfrey line, and let

$$N = \{(x, -x) : x \in \mathbb{Q}\}.$$

N is a closed subgroup of the paratopological (Abelian) group G, but the quotient paratopological group $G/_N$ is not Hausdorff.

1.3 Main properties of paratopological groups

Theorem 1. *Let X be a compact Hausdorff paratopological group. Then the inverse operation in X is continuous and, therefore, X is a topological group.*

Proof.
Let e be the neutral element of X. Since group X is Hausdorff, the set $M = \{(x, y) \in X \times X\colon xy = e\}$ is closed in the Cartesian product $X \times X$. Now, let F be any closed subset of X, and $P = (X \times F) \cap M$. Then F and $X \times F$ are compact, P closed in $X \times F$, since M is closed, therefore, P is compact set. Now, $(x, y) \in P$ if and only if $y \in F$ and $xy = e$, that is, $x = y^{-1}$. So the image of P under the natural projection of $X \times X$ onto the first factor X is precisely F^{-1}. Since F is compact and the projection mappings is continuous, we conclude that F^{-1} is compact and closed in X. So the inverse operation in X is continuous. □

Theorem 2. *Suppose that X is a Hausdorff paratopological group. Then, for each compact subset F of X, the set F^{-1} is closed in X.*

To prove this theorem clearly, we need to show definition of ultrafilter

Definition 8. *Given a set X, an ultrafilter on X is a set U consisting of subsets of X such that*

i) $\emptyset \notin U$

ii) If $A \subset X, B \subset X, A \subset B$, and $A \in U$ then $B \in U$.

iii) If $A, B \in U$, then $A \cap B \in U$.

iv) If $A \subset X$, then either A or $X \setminus A$ is an element of U.

Proof. [Th.2]
Let x be in the closure of F^{-1}. There exists an ultrafilter ξ on F^{-1} converging to x. Then $\eta = \{P^{-1} \colon P \in \xi\}$ is an ultrafilter on F. Since F is compact, there exists a point $y \in F$ such that η converges to y. From the continuity of the multiplication in X (X is paratopological group) it follows that the family $\gamma = \{PP^{-1} \colon P \in \gamma\}$ converges to the point $z = xy$. From the other side, the neutral element e of X belongs to all elements of γ. Since X is Hausdorff space, we conclude that $z = e$, which implies that $x = y^{-1} \in F^{-1}$. Thus, F^{-1} is closed in X. □

Theorem 3. *If X is a locally compact Hausdorff paratopological group, then the inverse operation in it is continuous, that is, X is a topological group.*

To prove this theorem we need the following lemma.

Lemma 1. *Suppose that X is a semitopological group, $\{U_n \colon n \in \omega\}$ is a sequence of open neighbourhoods of the neutral element e of X, and $\{x_n \colon n \in \omega\}$ is a sequence of points in X such that $x_n \in U_n$, for each $n \in \omega$, and the next conditions are satisfied:*

i) $\overline{U^2_{n+1}} \subset U_n$ *for each* $n \in \omega$;

ii) the sequence $\{y_k \colon k \in \mathbb{N}\}$, where $y_k = x_1 \ldots x_k$, has an accumulation point y in X. Then there exists $k \in \omega$ such that $x^{-1}_{k+1} \in U_0$.

Proof.
Since yU_1 is a neighbourhood of y, there exists $k \in N$ s.t. $y_k \in yU_1$. Let put $z = y^{-1}_{k+1}y$. Then

$$x^{-1}_{k+1} = y^{-1}_{k+1}y_k \in y^{-1}_{k+1}yU_1 = zU_1$$

and z is an accumulation point of the sequence $\{y^{-1}_{k+1}y_m \colon m \in \mathbb{N}\}$, by the separate continuity of multiplication in X (X is the semitopological group). It follows from condition i) of the lemma that, for each $m > k + 2$,

$$y^{-1}_{k+1}y_m = x_{k+2} \ldots x_m \in U_{k+2} \ldots U_m \subset U_{k+1}.$$

Therefore, $z \in \overline{U_{k+1}} \subset U_k$, which implies that $x^{-1}_{k+1} \in zU_1 \subset U_kU_1 \subset U_0$. □

So now we can prove the theorem.

Proof. [Th.3]
We need to check the continuity of the inverse at the neutral element e of X. So let assume the contrary. Then we can find an open neighbourhood U of e such that for each open set V containing e, V^{-1} is not a subset of U. Using the regularity of X and the continuity of multiplication, we can define a sequence of open sets $\{U_n : n \in \omega\}$ in X satisfying condition i) of Lemma 1. Since X is locally compact, we can also assume that the closure of U_0 is compact and contained in U. Now, by the choice of U, we can find a point $x_n \in Un$ s.t. x_n^{-1} is not in U, for each $n \in \omega$. Put $y_k = x_1 \dots x_k$, for each $k \in \mathbb{N}$. Then it easily follows from condition i) that all elements y_k are in U_0. Since the closure of U_0 is compact, there exists an accumulation point y for the sequence $\{y_k : k \in N\}$ in X. Thus all conditions of Lemma 1 are satisfied; by using it, we obtain $k \in \omega$ such that $x_{k+1}^{-1} \in U_0$, contradicting $U_0 \subset U$ and $x_{k+1}^{-1} \in X \setminus U$. □

Theorem 4. *[A.V. Arhangelskii and E.A. Reznichenko]*
Suppose that G is a paratopological group such that G is a dense $G_\delta -$set in a regular feebly compact space X. Then G is a topological group.

Before we prove this theorem we need to introduce definition of $G_\delta -$set, feebly compact space and prove necessary lemmas.

Definition 9. *In a topological space a $G_\delta -$set is a countable intersection of open sets.*

Definition 10. *A topological space is feebly compact if every locally finite cover by nonempty open sets is finite.*

Lemma 2. *Suppose that G is a paratopological group, and U any open neighbourhood of the neutral element $e \in G$. Then $\overline{M} \subset MU^{-1}$, for each subset M of G.*

Proof.
Put $A = \{g \in G : gU \cap M = \emptyset\}$ and $F = G \setminus AU$. Then, clearly, F is a closed subset of G and $M \subset F$. Therefore, $\overline{M} \subset F$. Take any $y \in F$. Then $yU \cap M = \emptyset$, that is, $yh = m$, for some $h \in U$ and $m \in M$. Hence, $y = mh^{-1} \in MU^{-1}$. Thus, $F \subset MU^{-1}$. Since $\overline{M} \subset F$, it follows that $\overline{M} \subset MU^{-1}$. □

Lemma 3. *Suppose that G is a paratopological group and not a topological group. Then there exists an open neighbourhood U of the neutral element e of G such that $U \cap U^{-1}$ is nowhere dense in G, that is, the interior of the closure of $U \cap U^{-1}$ is empty.*

Proof.
The inverse operation in G is discontinuous. Therefore, it is discontinuous at e, and we can choose an open neighbourhood W of e such that $e \notin Int(W^{-1})$. Since multiplication is continuous in G, we can find an open neighbourhood U of e such that $U^3 \subset W$. We claim that the set $U \cap U^{-1}$ is nowhere dense in G. Assume the contrary. Then there exists a non-empty open set V in G such that $V \subset U \cap U^{-1}$. From Lemma 2 it follows that $V \subset U \cap U^{-1} \subset (U \cap U^{-1})U^{-1} \subset U^{-2}$. Then $VU^{-1} \subset U^{-3} \subset W^{-1}$. So $V \cap U = \emptyset$, and the set VU^{-1} is open in G. Therefore, $e \in VU^{-1} \subset Int(W^{-1})$. So we have a contradiction. □

So now we can start prove our theorem.

Proof. [Th.4]
Assume the contrary. Then, by lemma 3, there exists an open neighbourhood U of the neutral element e of G such that $U \cap U^{-1}$ is nowhere dense. Let W be an open neighbourhood of e such that $\overline{WW} \subset U$. Put $O = W \setminus \overline{U \cap U^{-1}}$. Then, clearly, $O \subset W \subset O$ and $O^{-1} \cap U = \emptyset$.

First, we fix a sequence $\{M_n \colon n \in \omega\}$ of open sets in X such that $G = \bigcap_{n=0}^{\infty} M_n$ .We are going to define a sequence $\{U_n \colon n \in \omega\}$ of open subsets of X and a sequence $\{x_n \colon n \in \omega\}$ of elements of G such that $x_n \in U_n$, for each $n \in \omega$. Put $U_0 = O$, and pick a point $x_0 \in U_0 \cap G$.

Assume now that, for some $n \in \omega$, an open subset U_n of X and a point $x_n \in U_n \cap G$ are already defined. Since $e \in W \subset \overline{O}$, we have $x_n \in x_n\overline{O} = x_n\overline{O}$. Since U_n is an open neighbourhood of x_n, it follows that $U_n \cap x_nO = \emptyset$. We take x_{n+1} to be any point of $U_n \cap x_nO$. Note that $x_{n+1} \in G$, since $x_nO \in G$.

Using the regularity of X, we can find an open neighbourhood U_{n+1} of x_{n+1} in X such that the closure of U_{n+1} is contained in $U_n \cap M_n$, and $U_{n+1} \cap G \subset x_nO$. The definition of the sets U_n and points x_n, for each $n \in \omega$, is complete. Note that $\overline{U_i} \subset Uj$ whenever $j < i$. We also have $x_{n+1} \in x_nO$, for each $n \in \omega$.

Put $F = \bigcap_{n\in\omega} \overline{U_n}$. Clearly, $F \subset G$, and $F \neq \emptyset$ since X is feebly compact. The set FW is an open neighbourhood of F in G. Consider the closure P of FW in X, and let H be the closure of $X \setminus P$ in X. Then H is a regular closed subset of X, so that H is feebly compact.

We claim that $H \cap F = \emptyset$. Indeed, assume the contrary, and fix $x \in F \cap H$. Since FW is an open neighbourhood of F in G, from $x \in F$ it follows that there exists an open neighbourhood V of x in X such that $V \cap G \subset FW$. Then the density of G in X implies that $V \subset P$, while $x \in V \cap H$ implies that $V \setminus P = \emptyset$, which is a contradiction. Thus $H \cap F = \emptyset$.

Since H is feebly compact, our definition of F implies that $U_k \cap H = \emptyset$, for some $k \in \omega$ (we use that $\overline{U_i} \subset U_j$ whenever $j < i$). Then $U_k \subset P$. Since $x_k \in U_k \cap G$, it follows that $x_k \in \overline{FW}$. However, $F \subset U_{k+2} \cap G \subset x_{k+1}O \subset x_{k+1}W$. Hence, $x_k \in \overline{FW} \subset x_{k+1}\overline{WW} \subset x_{k+1}U$. Taking into account that $x_{k+1} \in x_kO$, we obtain that $x_k \in x_kOU$. Hence, $e \in OU$ and $O^{-1} \cap U = \emptyset$, which is again a contradiction □

Theorem 5. *A dense subgroup of a precompact paratopological group is precompact*

Proof. Let H be a dense subgroup of a precompact paratopological group G. Take an arbitrary neighborhood U of the neutral element e in G. First we show that H contains a finite subset F such that $G = UF$. Choose an open neighborhood V of e such that $V^2 \subset U$. Since G is precompact, there exists a finite subset C of G such that $G = VC$. For every $x \in C$, take an element $h_x \in H \cap x^{-1}V$ and put $F = \{h_x^{-1} \colon x \in C\}$. Clearly F is a finite subset of H. It follows from $h_x \in x^{-1}V$ that $x \in Vh^{-1}x$, hence $V_x \subset V^2h^{-1}x \subset Uh^{-1}x$, for each $x \in C$. We conclude, therefore, that $G = VC \subset UF$, i.e., $UF = G$. Finally, suppose that W is an open neighborhood of e in H. Take an open set U in G such that $W = U \cap H$. We have just shown that H contains a finite set F such that $G = UF$. Since H is a subgroup of G, our choice of U implies that $H = WF$. A similar argument shows that H contains a finite subset F' such that $H = F'W$. Hence H is precompact. □

Theorem 6. *Let G be a paratopological group, F be a compact subset of G, and P be a closed subset of G such that $F \cap P = \emptyset$. Then there exists an open neighbourhood V of the neutral element e such that $FV \cap P = \emptyset$ and $VF \cap P = \emptyset$.*

Proof.

Since the left translations in group G are continuous, we can choose, for every x from F, an open neighbourhood V_x of the neutral element e in G such that $xV_x \cap P = \emptyset$. Using the joint continuity of the multiplication in G, we can also take an open neighbourhood W_x of element e such that $W_x^2 \subset Vx$. The open sets xW_x, with $x \in F$, cover the compact set F, so there exists a finite set $C \subset F$ such that $F \subset \bigcup_{x\in C} xW_x$. Put $V_1 = \bigcap_{x\in C} W_x$. We claim that $FV_1 \cap P = \emptyset$. Indeed, it suffices to verify that

$yV_1 \cap P = \emptyset$, for each $y \in F$. Given an element $y \in F$, we can find $x \in C$ such that $y \in xW_x$. Then $yV_1 \subset xW_xV_1 \subset xW_xW_x \subset xV_x \subset G \setminus P$, by our choice of the sets V_x and W_x. This proves the fact that the sets FV_1 and P are disjoint. Similarly, one can find an open neighbourhood V_2 of e in G satisfying $V_2F \cap P = \emptyset$. Then the set $V = V_1 \cap V_2$ is as required and our prove are finished. □

Proposition 1. *For any two compact subsets E and F of a paratopological group G, their product EF in G is a compact subspace of G.*

Proof.
Proof of this fact is very simple. Since multiplication in a paratopological group is jointly continuous, the subspace EF of G is a continuous image of the Cartesian product $E \times F$ of the spaces E and F. Since EF is compact, by Tychonoffs theorem, the space EF is also compact. □

1.4 Exercises

Below we present few exercises, connected with paratopological groups.

1. Suppose that H is a dense subgroup of a paratopological group G with identity e. Show that if H is commutative, then so is G. Verify that if $n \in \mathbb{N}$ and every element $x \in H$ satisfies $xn = e$, then all elements of G satisfy the same equation.

2. Prove that for every element g of a Hausdorff paratopological group G, the set

$$G_g = \{x \in G \colon xg = gx\}$$

   is a closed subgroup of G. Show that Gg need not be invariant in G, even if G is a topological group.

3. Let D be a dense subset of a topological group G. Verify that the equalities $UD = G = DU$ hold for every neighbourhood U of the neutral element in G. Is the conclusion valid for paratopological (or quasitopological) groups?

4. Let G be an abstract group and n a positive integer. Prove that if τ is a Hausdorff paratopological group topology on G, then the set $G[n] = \{x \in G : xn = eG\}$ is closed in (G, τ). Does the conclusion remain valid for semitopological (quasitopological) group topologies on G?

We will try to solve two of those exercices.

- Ad.3
  It is known that G_g is (Hausdorf paratopological) subgroup of G for every $g \in G^3$. We will show that G_g is closed. Consider continuous translations λ_g and ϱ_g. Then $G_g = \{x \in G \colon \lambda_g(x) = \varrho_g(x)\}$. We need a small lemmas to proceed.

  Lemma. *A topological space X is Hausdorff if and only if the diagonal*

$$\Delta = \{(x, x) \in X \times X \colon x \in X\}$$

  is closed.

Proof.
Assume that X is Hausdorff. Now pick $(x, y) \in X \times X$ such that $x \neq y$. There exists a open set U and V such that $x \in U, y \in V$ and $U \cap V \neq \emptyset$. Then $U \times V$ is open in $X \times X$ and $(x, y) \in U \times V$. Clearly Δ has an empty intersection with $U \times V$. Then every point $(X \times X) \setminus \Delta$ lies with an open neighbourhood. The complement of Δ is open, which shows that Δ is closed. The converse will not be used in this exercise and the proof will be omitted. □

Lemma. *Let $f, g\colon X \to Y$ are two continuous functions between X and Y where Y is Hausdoff. The equalizer $\{x \in X : f(x) = g(x)\}$ is closed on X.*

Proof.
Consider a map $E\colon X \times Y \times Y$ defined as $E(x, y) = (f(x), g(x))$, E is clearly continuous. The equalizer is a preimage of $\Delta = \{(y, y) \in Y \times Y : y \in Y\}$ under E, that means
$\{x \in X : f(x) = g(x)\} = E^{-1}(\Delta)$ □

From the previous lemma Δ is closed as Y is Hausdorf and then the equalizer is closed. Since G_g is equalizer of λ_g and ϱ_g, this subgroup is closed. For a counterexample, consider the dihedral group of order 6 (which is isomorphic to $S3$). Then for any symmetry $s \in S3$ we have $G_s = \{x \in S3\colon sx = xs\} = \{e, s\}$ which fails to be an invariant group.

- Ad.4
  Let D be a dense subset of a topological group G. Verify that the equalities $UD = G = DU$ hold for every neighbourhood U of the neutral element in G. UD can be described as

$$UD = \{ud\colon u \in U, d \in D\} = \bigcup_{d \in D} \{ud\colon u \in U\} = \bigcup_{d \in D} Ud = \bigcup_{d \in D} \varrho_d(U)$$

In a semitopological group for each $x \in D$ the right translation the ϱ_x is an open map. It remains to show that for every $x \in G$ there exists $d \in D$ such that $x \in Ud$. In quasitopological group if D is dense, then so is D^{-1}. Pick then $d^{-1} \in D^{-1}$ such that $d^{-1} \in x^{-1}U$. It is equivalent to the statement that $x \in Ud$ and therefore the main statement becomes proven for quasitopological groups.

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